Optimal Control of an Elastic Tyre-Damper System with Road Contact

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We study an elastic tyre with a wheel rim that is suspended at the chassis of a car by means of a spring-damper element. This quarter car model may be controlled by varying the damping constant of the electrorheological damper. Our mathematical model yields a coupled ODE-PDE problem with a free boundary at the tyre-road contact. In this study we approximate the tyre by the Hertz contact stress formula. The resulting optimal control problem with control constraints is solved numerically.

1 Modelling of a quarter car and differential equations

In modern cars a dynamic control of the vehicle suspension is employed for better safety and comfort. The road profile is monitored, e.g. by a stereo camera, and the suspension is actively controlled. We focus on a proactive control of an electrorheological damper where the damping constant $D \in J = [D_{min}, D_{max}]$ may be varied. In our model we do not use additional actuator forces for the control. We consider an elastic tyre-damper system where the wheel rim is connected to the car chassis by a spring-damper element. The tyre has a contact boundary with the road where it is deformed, the free boundary and the deformation depending on the weight of the car and the elastic forces. For simplicity, we consider a 2d cross-section of the quarter car model. The geometry of the model is depicted in Fig. 1, the time-interval is $(0, T)$.

The system is modelled by an ordinary differential equation (ODE) for the spring-damper-system, the stationary elasticity equation for the tyre deformation, and a complementarity condition for the free boundary with the road. Newton's law of motion yields for the (relative) displacement z of the spring from its rest position (e.g. [1, (4.1)])

$$m\ddot{z} = -D(\dot{z} - \dot{y}) - k(z - y) \ \text{ in } (0, T), \quad z(t = 0) = \dot{z}(t = 0) = 0 \tag{1}$$

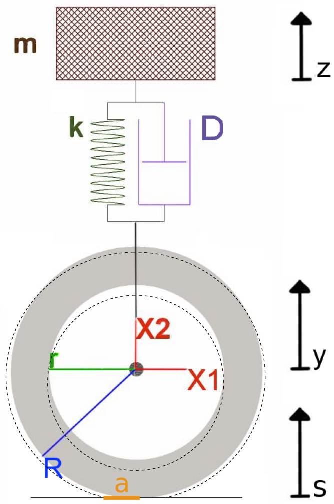

Fig. 1: Geometry of the quarter car model with free road contact. The elastic tyre is described in the undeformed configuration (solid grey).

where y is the (relative) displacement of the rim, m the quarter mass of a car chassis, and k the spring constant. As an approximation the elastic tyre is modelled in linear elasticity yielding the partial differential equation (PDE) for the displacement $u(x, t)$ in Lagrangian coordinates (undeformed configuration) with boundary conditions (b. c.):

$$-\operatorname{div}\sigma(u) = -\rho g e_2 \qquad \text{in } \{r < (x_1^2 + x_2^2)^{1/2} < R\} \times [0, T], \tag{2}$$

$$u = y(t)\, e_2 \ \text{ or } \ -\sigma(u).n = -F(t)/(\pi r b) e_2 \qquad \text{on } \{(x_1^2 + x_2^2)^{1/2} = r\} \times [0, T], \tag{3}$$

$$0 \leq -n \cdot \sigma(u).n \perp ((s(x_1, t) - R + x_2)\, e_2 - u) \cdot n \geq 0 \qquad \text{on } \{(x_1^2 + x_2^2)^{1/2} = R\} \times [0, T], \tag{4}$$

$$\tau \cdot \sigma(u).n = 0 \qquad \text{on } \{(x_1^2 + x_2^2)^{1/2} = R\} \times [0, T]. \tag{5}$$

This is a Signorini problem. Here σ denotes the Cauchy stress tensor, linear in ∇u, depending on the Young modulus E and the Poisson ratio ν. ρ prescribes the mass density within the tyre and g the gravitational acceleration. Eq. $(3)_1$ models that the displacement u is continuous at the wheel rim where the (stiff) wheel rim is shifted by the unknown y, while the second b.c. models the balance with the force F, exerted by the rim, per area (b the width of the tyre). In general both b.c. in (3) have to be satisfied, one b.c. enters the PDE problem for u, the other determines y. Eq. (4) & (5) encodes that there is either a negligible outer pressure and a positive gap between the deformed tyre and a given road profile s, or there is a positive contact pressure and contact between deformed tyre and road. n denotes the outer normal of the tyre, τ the tangential.

The symmetric free boundary is determined approximately by the Hertz formula (plain strain) [4] yielding $a = \sqrt{Rd}$ with the maximal penetration depth $d = -\frac{4}{\pi}\frac{1-\nu^2}{Eb}F$ under the static force $F = -(m + m_T)g + k(z - y)$ where m_T the mass of tyre and rim. On the contact boundary $\Gamma_c := \partial B_R(0) \cap \{|x_1| < a(t), x_2 < -r\}$, according to Hertz for the normal pressure $-e_2 \cdot \sigma(u).e_2 = (\frac{E}{1-\nu^2}\frac{F}{bR}((x_1/a)^2 - 1))^{1/2} \geq 0$, since typically $F \ll 0$. The Hertz approximation is justified numerically by

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solving the complementarity conditions iteratively [3]. The approximation motivates to replace (4) & (5) by the explicit b.c.

$$-\sigma(u).n = 0 \text{ on } (\partial B_R(0) \setminus \Gamma_c) \times (0,T), \quad u = \left(s(x_1,t) - R + (R^2 - x_1^2)^{1/2} \right) e_2 \text{ on } \Gamma_c \times (0,T). \tag{6}$$

If we model the tyre as a second spring [1, (4.2)], then we have $m_T \ddot{y} = -(m+m_T)g + k(z-y) + D(\dot{z}-\dot{y}) - k_T(y - s(0,\cdot))$, where we introduce k_T as a "spring" constant for the tyre. As a first approach, the vertical acceleration and the damping term are neglected, motivated by $m_T \ll m$ and $D \ll k$. In $(3)_1$, we approximate $u_2|_{\partial B_r(0)} - s(0,\cdot) \approx -d$ which is reasonable for small strains. We identify $k_T = \frac{\pi}{4}\frac{Eb}{1-\nu^2}$ and we end up with the explicit formula $y = \frac{1}{k_T+k}(-(m+m_T)g + kz + k_T s(0,\cdot))$.

2 Optimal control problem and numerical results

The goal is to minimize the objective $I(D) = \frac{1}{2}\int_0^T \alpha_1|\ddot{z}(D)|^2 + \alpha_2|z(D)-y(D)|^2 + \alpha_3|y(D)-s(0,\cdot)|^2$ where the parameters α_k, $k = 1,2,3$, are positive. Here we consider the states u, z, and y as functions of the control D. Minimizing the first term in the reduced objective function corresponds to an increase in the comfort, the second term accounts for the spring robustness (a spring cannot be contracted/extended without limits), and the last "safety" term models the change of distance between rim and road (neither the grip i.e. the contact area should go to zero nor the chassis should touch the road) [1, 4.3.1.].

As a first approach we solve the optimal control problem to minimize $I(D)$ subject to (1), (2), $(3)_1$, (6), and $D \in J$ by a projected gradient method with Armijo line search. The gradient is computed by a sensitivity based approach. Within our approximation u does not enter into the reduced cost functional and there is no need to solve a sensitivity PDE for $u_{,D}$. Besides $y_{,D} = \gamma z_{,D}$ where $\gamma = k/(k_T + k)$. For the sensitivity $z_{,D}$ we have the ODE $m\ddot{z}_{,D} = -(1-\gamma)\dot{z} - D(1-\gamma)\dot{z}_{,D} - k(1-\gamma)z_{,D}$. The ODE for z and $z_{,D}$ are discretized by Heun's method and the PDE for u may be solved by FEM. This is implemented in the open software FEniCS. As a stopping condition we work with $(I(D)^{(i)} - I(D)^{(i+1)})/(1 + |I(D)^{(i+1)}|) < err$ where the index (i) indicates the optimization iteration. For numerical results, in case of an initial control $D^{(0)} = 1500$, weights $\alpha_1 = 1/4$, $\alpha_2 = 10^2$, $\alpha_3 = 10^4$, $err = 10^{-13}$, and $2 \cdot 10^4$ time steps, see Fig. 2. The given road profile is $s(x_1,t) = 0.25(1 + \cos(2(x_1+t)))\,\chi_{[\pi/2,3\pi/2]}(x_1+t)$, χ a characteristic function. Realistic data for a midsized car has been used. We observe an antagonistic behaviour between optimal comfort and safety as in [5]. Comfort yields high oscillations of D, while safety requires high accelerations $\ddot{z}$.

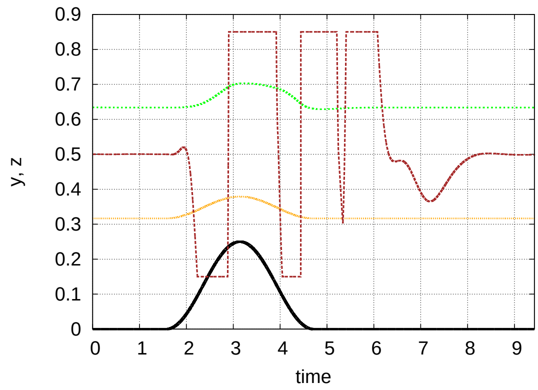

Fig. 2: Computed optimal control D (brown dashed line) for a compromise between comfort and safety, $T = 3\pi$. Relative displacement z of the chassis (shifted by $2R$, green upper dotted line) and y of the rim (shifted by R, orange lower dotted line) together with cosinusoidal road profile $s(0,t)$ modelling a speed bump (black continuous line).

The approach to optimize the vehicle dynamics by means of optimal control and not only by feedback control seems to be new within this field [5]. In many other models the elastic tyre is replaced at once by a spring [1, 2, 5] or several springs (e.g. software CDTire by ITWM Kaiserslautern). These models might, as well as our model, allow for a real-time control which is out of sight for a fully ODE-PDE model. However, in these analogous models the "tyre spring" parameters have to be fitted, that are given directly in our model. Compared to [3], where another approximation $F \approx -(m+m_T)g$ is used, we obtain slight corrections. Well-posedness of this problem will be discussed in an upcoming study. A future goal is to work without the Hertz approximation and to consider the full coupling between u and y. This situation raises also interesting analytical questions. Finally, it might be interesting to incorporate friction and non-linear elasticity into our model.

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References

[1] B. Cai, Neural Networks, Fuzzy Logic, and Optimal Control for Vehicle Active Systems with Four-Wheel Steering and Active Suspension, Fortschrittsberichte VDI Reihe 12 Nr. 196 (VDI Verlag, Düsseldorf, 1993).

[2] J. Michael and M. Gerdts, Optimal Control in Proactive Chassis Dynamics - A fixed step size time-stepping scheme for complementarity problems. In: Progress in Industrial Mathematics at ECMI 2012, Mathematics in Industry, Vol. 19, Springer, 2014. To appear.

[3] R. Moritz, Simulation und Optimierung eines Reifens in Fahrbahnkontakt. Master thesis (Universität der Bundeswehr München, Neubiberg, 2013).

[4] V. L. Popov, Contact Mechanics and Friction (Springer, Heidelberg, 2010).

[5] U. Rettig, and O. von Stryk, Optimal and Robust Damping Control for Semi-Active Vehicle Suspension. In: Proc. 5th EUROMECH Nonlinear Dynamics Conference (ENOC 2005), Eindhoven, The Netherlands (2005) pp. paper no. 20-316.