Modelling, simulation and optimization of an elastic structure under moving loads

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We consider an elastic structure that is subject to moving loads representing e.g. heavy trucks on a bridge or a trolley on a crane beam. A model for the quasi-static mechanical behaviour of the structure is derived, yielding a coupled problem involving partial differential equations (PDE) and ordinary differential equations (ODE). The problem is simulated numerically and validated by comparison with a standard formula used in engineering. We derive an optimal policy for passing over potentially fragile bridges. In general, our problem class leads to optimal control problems subject to coupled ODE and PDE.

The elastic structure is modelled by a single solid thin cuboid i.e. a beam in three dimensions. By choosing suitable boundary conditions (b.c.) this serves as a simple model either for a bridge, fixed on both ends of the solid beam, or for a crane, fixed only at one side. The bridge could be passed over by vehicles that are modelled as several area loads. In case of a crane, the goal is to transport a load from an initial to a terminal position, where the load is fixed to a moving trolley that runs along the crane beam and that is modelled as well as an area load. An objective could be to pass the bridge or to move the trolley in minimal time, while the structure should not be damaged, e.g. the elastic deflection is to be minimized. A motivation for the bridge problem is the damage caused mainly by heavy trucks to road bridges and overcrossings. Heavy traffic is one of the reasons for increasing road maintenance costs. An application for the crane model is that it serves as a first step to the challenging control and optimal control for pontoon cranes that would require enhanced models. Both situations share some mathematical aspects that are discussed in this short paper together. The investigation of the simple model for the bridge has been started in [1,4]. The optimal control problem for an elastic crane with a moving load is considered in [2] in details.

As geometry we consider the undeformed reference configuration $\Omega = \{x \in \mathbb{R}^3 \,|\, 0 \le x_1 \le \ell, |x_2| \le b, |x_3| \le h\}$ in space and the time interval is $[0,T]$. In this short study the terminal time $T > 0$ is fixed. The mechanical displacement field $u : \Omega \to \mathbb{R}^3$ is considered in (undeformed) Lagrangian coordinates. The other states q correspond to the centres of mass of moving loads. The time-dependent control $U : [0,T] \to \mathbb{R}$ enters the state equations for the loads. U is subject to control constraints, i.e. $U(t) \in [U_{min}, U_{max}]$ for all times $t \in [0,T]$ for given $U_{min} < 0 < U_{max}$, modelling maximal deceleration and acceleration, respectively. For a bridge the structure is clamped at both ends, $\Gamma_D = \{x \in \partial\Omega \,|\, x_1 = 0 \vee x_1 = \ell\}$, while for a crane the beam is clamped at one end only, i.e. $\Gamma_D = \{x \in \partial\Omega \,|\, x_1 = 0\}$. The remaining boundary $\Gamma_N := \partial\Omega \setminus \overline{\Gamma_D}$ is subject to Neumann boundary conditions. In linear elasticity, i.e. under the assumption of small displacement gradients $\|u\| \ll 1$, the mechanical strain reads $\epsilon(\nabla u) = (\nabla u + \nabla u^\top)/2$. The stress is modelled by the Cauchy stress tensor $\sigma(\nabla u) = E\tilde{\nu}/((1+\tilde{\nu})(1-2\tilde{\nu}))\,\mathrm{trace}(\nabla u)\, Id_3 + E/(1+\tilde{\nu})\,\epsilon(\nabla u)$, depending on the Young modulus $E > 0$ and the Poisson number $\tilde{\nu} \in (-1, 1/2)$. This yields the following quasi-static PDE completed with Dirichlet and Neumann boundary conditions:

$$-\mathrm{div}\,\sigma(\nabla u) = H \qquad \text{in } \Omega \times [0,T], \tag{1}$$

$$u = 0 \qquad \text{on } \Gamma_D \times [0,T], \qquad -\sigma(\nabla u).\nu - G(q,\bar{u},U) = 0 \qquad \text{on } \Gamma_N \times [0,T]. \tag{2}$$

Here $H = -\rho g e_3$ is a volume force (due to the dead load of the bridge/crane beam with density ρ, g being the gravity acceleration) and ν denotes the outer normal. $G : [0,T] \times \Gamma_N \to \mathbb{R}^3$ is a boundary force modelling the total area forces of N loads with mass m_i at position q_i ($i = 1,\ldots,N$) and has the following structure $G(q(t),\bar{u}(t),U(t),x) = \sum_{i=1}^n \chi_{\Gamma_C^i(q_i(t))}(x)\, G_0(q_i(t),\bar{u}(t),U(t),m_i)$, where $\chi_{\Gamma_C^i(q_i(t))}(x) : \Gamma_N \to \{0;1\}$ is the characteristic function of the contact area Γ_C^i, $i = 1,\ldots,n$ ($n \le N$), where the area force G_0 is applied. The contact area is shifting with time: $\Gamma_C^i(q_i(t)) := \Gamma_C^{i,0} + q_i(t)$. Note that in G and G_0 a vector of mean values $\bar{u}$, averaging spatially the functions u, ∇u, and $D^2 u$ over the contact area Γ_C^i enters. The ODE states $q : [0,T] \to \mathbb{R}^N$ are subject to the Newton law of motion $M(q)\,\ddot{q} = F(q,\dot{q},\bar{u},U)$ for all times $t \in [0,T]$, where M is a given mass matrix and F is a generalized force, combining Coriolis and external forces, the latter depending on the control U. Since it is reasonable to consider a model, where the ODEs do not depend additionally on x, this motivates that the dependence of F on u is modelled by averaging over u on Γ_C^i and considering a dependence of F on $\bar{u}$ instead. Terminal conditions for the ODE states could be added as quadratic penalty terms to the objective $J(U,u,q,\dot{q})$ that is to be minimized w.r.t. U.

The PDE problem (1) & (2) is solved by the finite element method, while the ODEs are discretized by an Euler method. In order to avoid locking effects (typical for thin beam-like structures) and to represent the second order derivatives D^2u, we

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consider quadratic Lagrange elements. The mesh is refined on the part of Γ_N, where the loads may possibly move. In order to avoid technical problems, whether a finite cell belongs to Γ_C^i or not, the characteristic functions $\chi_{\Gamma_C^i}$ are approximated smoothly. The dependence of the b.c. (2).2 on mean values $\bar{u}$ over u is solved by an inner fixed point loop. An outer loop, being another fixed point iteration, is applied for solving the possibly fully coupled ODE-PDE system. For details for the algorithm see in case of the crane [2, Sect. 3 & 4].

The algorithms for the bridge as well as for the crane problem have been implemented in the free, open source finite element library FEniCS. Here we present the simple case of trucks, passing with constant velocity over the whole bridge. This situation has been simulated for a full bridge (3-dimensional beam) and optimized by comparing various scenarios (number of trucks, distance, opposing traffic) in [4]. Here the simulations are validated by increasing the polynomial degree of trial functions and comparing maximal displacements of the bridge. In this case the maximum is attained at the centre of the bridge. Varying parameters, we observe that the width b of bridge has almost no impact on the maximal displacement, while length ℓ and height h do. This behaviour is due to the planar moment of inertia $I_3 = bh^3/12$ [5, §17]. For the simulation of two trucks following each other at two different distances, see Fig. 1. The opposing traffic of trucks exhibits even more critical effects.

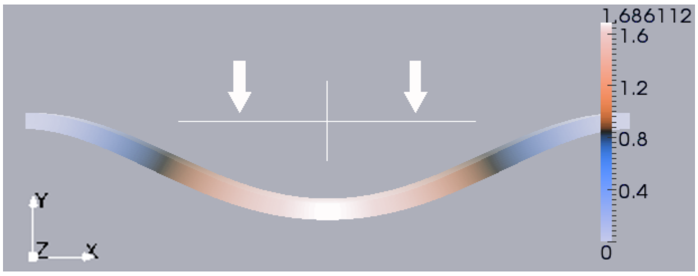

Fig. 1: Deformed beam (scaled), colored according to absolut deformation $\|u\|$, in the case of two trucks following each other with a distance of 12 m, at $t = T/2 = 20$ s, when the maximal deflection is attained at the centre of the bridge. Here $\ell = 100$ m, $b = 6$ m, $h = 3$ m, $E = 2.1 \cdot 10^6$ N/m^2, $\tilde{\nu} = 0.3$, $\rho = 7850$ kg/m^2, and $G_0 = -10^7\, e_3$ N/m^2 acts for both trucks on an area $\Gamma_C^0 = 3\text{m}\times 18\text{m}$. The two centres of mass are indicated by arrows, cf. [4, Abb. 23].

However, the simulation of a full 3-dimensional model of the bridge involves many unknowns due to the 3-dimensional mesh. This motivates to reduce the model by replacing the full bridge structure by a suitable plate model. For modelling the bridge as a Kirchhoff plate and the corresponding simulations see [1]. Then the structure is reduced to a 2-dimensional geometry $\tilde{\Omega} = \{x \in \Omega \,|\, x_3 = 0\}$. The resulting PDE in $\tilde{\Omega} \times [0, T]$ for the 2-dimensional displacement w is the Kirchhoff plate equation $k\, \Delta^2 w = \tilde{G}$, where k is the rigidity constant of the plate and $\tilde{G}$ represents the original effects of volume and area forces, H and G. It is an elliptic forth-order PDE and it is equipped with two suitable b.c., depending on whether the plate is fully clamped or simply supported at the left- and right-hand-sides, respectively. The implementation uses a discontinuous Galerkin method using a penalty term in order to guarantee continuity over element boundaries [6]. The simulation [1] is validated by comparing the maximal deflection in vertical direction with a standard formula [5, §20]. For the optimal control of the elastic crane-trolley-load system by means of a sensitivity-based first-discretize-then-optimize approach, see [2].

Our simulations suggest as an optimal policy for driving over the bridge, to respect a certain minimal distance between two heavy vehicles and, in particular, to avoid opposite traffic. This shows that our simple model for an elastic bridge verifies statements well-known in civil engineering. However, so far our models use the quasi-static PDE of linear elasticity and cannot prescribe a swinging behavior due to elastic waves that might be relevant e.g. for large suspension bridges. The latter would require second-order time derivatives of u in (1), leading to a hyperbolic PDE instead of an elliptic PDE. The overall aim is an optimal control for passing over potentially fragile bridges. This leads to an optimal control problem subject to coupled ODE and PDE as considered for the elastic crane-trolley-load system in [2]. For an efficient computation of this full optimal control problem for the bridge, it could be helpful to study, whether the bridge model could be reduced further to a 1-dimensional Euler-Bernoulli beam. So far only a few studies exist for optimal control of fully coupled ODE-PDE, see e.g. the optimal braking of a truck with a fluid container [3] and the references therein.

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