The finite element method with anisotropic mesh grading for the Poisson problem in domains with edges*

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Abstract. This paper is concerned with the anisotropic singular behaviour of the solution of elliptic boundary value problems near edges. The paper deals first with the description of the analytic properties of the solution. The finite element method with anisotropic, graded meshes and piecewise linear shape functions is then investigated for such problems; the schemes exhibit optimal convergence rates with decreasing mesh size. For the proof, new local interpolation error estimates in anisotropically weighted spaces are derived. Finally, a numerical experiment is described, that shows a good agreement of the calculated approximation orders with the theoretically predicted ones.

Key Words. Elliptic boundary value problem, singularities, finite element method, anisotropic mesh grading.

AMS(MOS) subject classification. 65N30, 35D10

1 Introduction

Consider the Poisson problem with in general mixed boundary conditions in a three-dimensional polyhedral domain. We are interested in situations where the solution has singular behaviour near edges, for example in the Dirichlet problem when the interior angle at some edge is greater than π. The anisotropic structure of the edge is reflected by an anisotropic behaviour of the solution near the edge: The singular part of the solution can be represented by a convolution of some two-dimensional singularity functions with a regular function in the third direction, see Section 2.

It is well known that these singularities lead to a low approximation order of the standard finite element method. Two-dimensional problems with corner singularities can be treated with certain mesh refinement near these corners in order to improve the approximation order [5, 6, 11, 13]. Thus it seems to be natural to treat edge singularities with meshes of tensor product form, graded perpendicularly to the edge and quasi-uniform in the edge direction. Pentahedral meshes seem to be natural, but each pentahedron can easily be divided into three tetrahedra.

Such meshes are anisotropic in the sense that elements in the refinement region have an aspect ratio which is growing to infinity for $h \to 0$, h is the global mesh size. In [1] it is shown for tetrahedral meshes that this strategy is successful, but under strong smoothness assumptions to the data. Our aim is to relax these assumptions which requests some refined considerations which shall be motivated now.

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*This is a shortened and corrected version of [4]; it is only available via ftp or WWW (`HREF="file://ftp.tu-chemnitz.de/pub/Local/mathematik/Apel/an1.ps.Z"`). Another short version, partially without proofs, appeared in the Pierre Grisvard Memorial volume *Partial Differential Equations and Functional Analysis*, Birkhäuser, Boston, 1996.

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The investigation of the finite element error $u - u_h$ in the energy norm (here equivalent to the $W^{1,2}(\Omega)$-norm) is usually reduced via Céa's lemma to a general approximation problem. If we want to take advantage of anisotropic finite element meshes, we need an approximation operator for which error estimates are available that take these different asymptotic mesh sizes of the elements into account. Here we rely on the Lagrangian interpolation operator because such estimates have already been derived [1]. Moreover, we note that Clement's operator [7] is not applicable in the anisotropic context because the $W^{1,2}(\Omega)$-error is in general not bounded by a $W^{2,p}(\Omega)$-norm of the solution. The same is true for the Scott/Zhang operator [14] in its general definition. There may be some hope that a modification of the latter operator may be suited for anisotropic elements but this is not clear to date.

The global interpolation error estimate is usually proved by taking the sum over elementwise error estimates. Here, we have to distinguish mainly two cases, namely elements at the edge and elements away from the edge (but within the refinement zone). For the latter ones one can exploit that $u \in W^{2,p}(\Omega_i)$ for some p, and use the local interpolation error estimate of [1]:

$$|u - u_h; W^{1,p}(\Omega_i)| \leq C \sum_{j=1}^{2} h_i \left| \frac{\partial u}{\partial x_j}; W^{1,p}(\Omega_i) \right| + Ch \left| \frac{\partial u}{\partial x_3}; W^{1,p}(\Omega_i) \right|. \tag{1.1}$$

Unfortunately, it holds true only for $p > 2$. This forced us to consider the regularity of u in Banach spaces. The diameters h_i of the finite element Ω_i, perpendicularly to the edge, are expressed by the product of the global mesh size h and some power r_i^β ($\beta \geq 0$) of the distance r_i of the element Ω_i to the edge considered. This leads to an estimate of the form

$$|u - u_h; W^{1,p}(\Omega_i)| \leq Ch \left(\sum_{i,j=1}^{2} \left\| r^\beta \frac{\partial^2 u}{\partial x_i \partial x_j}; L^p(\Omega_i) \right\| + \sum_{j=1}^{3} \left\| \frac{\partial^2 u}{\partial x_i \partial x_3}; L^p(\Omega_i) \right\| \right).$$

We define the expression in the parentheses to be a seminorm in some Banach space $A_\beta^{2,p}$ and have to investigate the question for which β the right hand side is bounded:

$$\|u; A_\beta^{2,p}(\Omega)\| \leq C \|f; L^p(\Omega)\|.$$

For this we used Grisvard's representation formula for u [10] and calculated the norm, see Section 2.

We encounter another problem by considering the elements Ω_i at the edge. Here the estimate (1.1) is not applicable due to $u \notin W^{2,p}(\Omega_i)$. Because anisotropic interpolation error estimates do not hold for functions in Sobolev-Slobodetskiĭ spaces (the seminorm can not be transformed in the desired way from the reference element to Ω_i), we extend (1.1) slightly for functions in weighted spaces $A_\beta^{2,p}(\Omega_i)$. This is quite straightforward, see Subsection 3.2. With all these ingredients we can finish the global error estimate.

In a last section we consider some other aspects: Because we treated for simplicity in Sections 2 and 3 the Dirichlet problem only, we discuss in Subsection 4.1 other boundary conditions.

For test calculations we can refer to another paper. In [3] we documented a test, where one problem was calculated with isotropic as well as with anisotropic graded meshes. We derived approximation orders from the finite element errors for different mesh size parameters h. We observed a good agreement of the calculated approximation orders with the expected ones (see (3.21)). Moreover, it turned out that the same error level can be achieved with less computational effort (smaller number of elements, of nodes, of degrees of freedom) with anisotropic meshes in comparison with isotropic ones. Another test is documented in Subsection 4.2. There, the exact solution has a jump in the second derivative in edge direction.

2 Analytical properties of the solution

Consider the Dirichlet problem for the Poisson equation,

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

over a bounded polyhedral domain $\Omega \subset \mathbb{R}^3$. In particular, we will focus on prismatic domains

$$\Omega = G \times I, \tag{2.2}$$

where $G \subset \mathbb{R}^2$ is a polygonal domain and $I = \,]0, z_0[\, \subset \mathbb{R}^1$ is an interval. The domain G has only one corner with interior angle $\omega > \pi$ at the origin; thus Ω has one "singular edge" which is part of the x_3-axis. The case of more than one singular edge can be treated similarly because the edge singularities we are interested in are of local nature. The restriction to prismatic domains is made because we want to consider edge singularities here, and such domains do not introduce additional corner singularities [15].

For general polyhedral domains we have to distinguish between corner and edge singularities. The corner singularities are not a problem of anisotropy; they can be treated with isotropic, graded meshes as introduced for example in [5]. The main problem is to construct meshes which are both anisotropic near edges and isotropic near those corners which cause singularities. We will discuss this in a forthcoming paper.

The variational form of problem (2.1) is given by:

$$\textit{Find } u \in \overset{o}{H}{}^1(\Omega) \textit{ such that } a(u,v) = (f,v) \textit{ for all } v \in \overset{o}{H}{}^1(\Omega). \tag{2.3}$$

The bilinear form $a(.,.)$ and the linear form $(f,.)$ are defined by

$$a(u,v) := \int_\Omega \sum_{i=1}^{3} \partial_i u \partial_i v \, d\underline{x}, \qquad (f,v) := \int_\Omega f v \, d\underline{x}.$$

We use the abbreviations ∂_i for $\frac{\partial}{\partial x_i}$ and ∂_{ij} for $\partial_i\partial_j$. The space $\overset{o}{H}{}^1(\Omega)$ is defined, as usual, by $\overset{o}{H}{}^1(\Omega) := \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. For the data we consider $f \in L^p(\Omega)$, $p \geq 2$. $L^p(.)$ $(1 \leq p \leq \infty)$ are the usual Lebesgue spaces, $W^{s,p}(.)$ $(s \geq 0,\ 1 \leq p \leq \infty)$ the Sobolev(-Slobodetskiĭ) spaces (sometimes we write $W^{0,p}(.)$ for $L^p(.)$), and $H^s(.) := W^{s,2}(.)$. — Note that the conditions of the Lax–Milgram lemma are satisfied; thus the solution $u \in \overset{o}{H}{}^1(\Omega)$ of problem (2.3) exists and is unique.

As motivated in the Introduction, we are interested for which β the solution u belongs to some anisotropically weighted space $A^{2,p}_\beta(\Omega) := \{v \in \mathcal{D}'(\Omega) : \|v; A^{2,p}_\beta(\Omega)\| < \infty\}$, with

$$\begin{aligned}
|v; A^{2,p}_\beta(\Omega)|^p &:= \int_\Omega \left\{ r^{\beta p} \sum_{i,j=1}^{2} |\partial_{ij} u|^p + \sum_{i=1}^{3} |\partial_{3i} u|^p \right\} d\underline{x}, \\
\|v; A^{2,p}_\beta(\Omega)\|^p &:= |v; A^{2,p}_\beta(\Omega)|^p + \int_\Omega \left\{ r^{(\beta-1)p} \sum_{i=1}^{2} |\partial_i u|^p + r^{-p} |\partial_3 u|^p + r^{(\beta-2)p} |u|^p \right\} d\underline{x},
\end{aligned}$$

and x_3 is the direction of the edge. The definition of the powers of r for the solution and its first derivatives were motivated by searching an optimal description of u. It is used for example in the proof of Theorem 3.5.

To treat this regularity problem we use a result of Grisvard [10] for a dihedral cone $D := C \times \mathbb{R}$, where C is an infinite cone of $\mathbb{R}^2$ of opening ω. We are concerned with the edge regularity of the variational solution $v \in \overset{o}{H}{}^1(D)$ of the Dirichlet problem

$$-\Delta v = g \in L^p(D), \tag{2.4}$$

for $p \geq 2$. Since we are only interested in the local behaviour of the solution, we suppose that v exists and has a compact support. As before, we denote by $\underline{x} = (x_1, x_2, x_3)$ the Cartesian coordinates in D, where $x_3 \in \mathbb{R}$ and $(x_1, x_2) \in C$, and by (r, φ) the polar coordinates in C. Let us recall this theorem [10, Theorem 6.6].

Theorem 2.1 *Suppose that $\frac{j\pi}{\omega} \neq 2 - \frac{2}{p}$ for all $j \in \mathbb{Z}$, then the solution $v \in \overset{o}{H}{}^1(D)$ of problem (2.4) admits the decomposition*

$$v = v_r + \sum_{0 < \frac{j\pi}{\omega} < 2 - \frac{2}{p}} (K_j \overset{x_3}{\star} q_j)\, \psi_j, \tag{2.5}$$

where $v_r \in W^{2,p}(D)$ is the regular part of v, $q_j \in B^{2-\frac{2}{p}-\frac{j\pi}{\omega},p}(\mathbb{R})$ (that means in the classical Sobolev space $W^{2-\frac{2}{p}-\frac{j\pi}{\omega},p}(\mathbb{R})$, if $2 - \frac{2}{p} - \frac{j\pi}{\omega} \notin \mathbb{Z}$, otherwise in the Besov space $B^{2-\frac{2}{p}-\frac{j\pi}{\omega},p}(\mathbb{R})$, see [16]), ψ_j are the 2D-singular functions of the Laplace operator in C:

$$\psi_j(r, \varphi) := \xi(r) r^{j\pi/\omega} \sin\left(\frac{j\pi\varphi}{\omega}\right), \tag{2.6}$$

and finally K_j are kernels defined by

$$\begin{aligned} K_j(r, x_3) &:= \frac{r}{\pi(r^2 + x_3^2)} \quad \text{if } \frac{j\pi}{\omega} > 1 - \frac{2}{p}, \\ K_j(r, x_3) &:= \frac{2r^3}{\pi(r^2 + x_3^2)^2} \quad \text{if } \frac{j\pi}{\omega} \leq 1 - \frac{2}{p}. \end{aligned}$$

There exists a positive constant C independent of g, such that

$$\|v_r; W^{2,p}(D)\| + \sum_{0 < \frac{j\pi}{\omega} < 2 - \frac{2}{p}} \|q_j; B^{2-\frac{2}{p}-\frac{j\pi}{\omega},p}(\mathbb{R})\| \leq C\, \|g; L^p(\mathbb{R})\|.$$

Here and in the sequel, $K \overset{x_3}{\star} q$ means the convolution with respect to the edge parameter x_3:

$$(K \overset{x_3}{\star} q)(r, x_3) := \int_{\mathbb{R}} K(r, s) q(x_3 - s)\, ds.$$

In view of that Theorem, it suffices to show that the 3D-singularity function

$$v_j := (K_j \overset{x_3}{\star} q_j)\psi_j \tag{2.7}$$

satisfies an inclusion of the type $v_j \in A^{2,p}_{\beta}(D)$. The proof is based on the next general result concerning convolution with arbitrary kernels, which is inspired from Theorem 6.5 of [10] (notice that this theorem had a different goal).

Lemma 2.2 *Let $K(r, x_3)$ be a kernel satisfying*

$$|K(r, x_3)| \leq C \frac{r^\beta}{(r^2 + x_3^2)^\gamma}, \forall r > 0, x_3 \in \mathbb{R}, \tag{2.8}$$

with some $C > 0$ and $\gamma > \frac{1}{2}$ (in order that K would be integrable with respect to x_3) and

$$\int_{\mathbb{R}} K(r, x_3)\, dx_3 = 0. \tag{2.9}$$

For $q \in B^{\sigma,p}(\mathbb{R})$, with $\sigma \in]0, 1]$, we set $h(r, x_3) := (K \overset{x_3}{\star} q)(r, x_3)$. If $\sigma < 2\gamma - 1$ and $\beta \geq -1 - \frac{2}{p} - \sigma + 2\gamma$, then there exists a constant $C_1 > 0$ (independent of q) such that

$$\left(\int_0^1 \int_{\mathbb{R}} |h(r, x_3)|^p\, r dr dx_3 \right)^{1/p} \leq C_1 \|q; B^{\sigma,p}(\mathbb{R})\|. \tag{2.10}$$

Proof From assumption (2.9), we may write $h(r, x_3) = \int_{\mathbb{R}} K(r, s)\{q(x_3 - s) - q(x_3)\}\, ds$, and taking the L^p-norm with respect to x_3, we obtain

$$\|h(r, x_3); L^p_{x_3}(\mathbb{R})\| \leq \int_{\mathbb{R}} |K(r, s)| \cdot \|q(x_3 - s) - q(x_3); L^p_{x_3}(\mathbb{R})\|\, ds. \tag{2.11}$$

Let us introduce the functions

$$\kappa(s) := |s|^{-\sigma-1/p}\, \|q(x_3 - s) - q(x_3); L^p_{x_3}(\mathbb{R})\|, \qquad k(t) := \frac{|t|^{2\gamma-\sigma-1}}{(1+t^2)^{\gamma}},$$

and the multiplicative convolution I of k with the function $s^{1/p}\kappa$,

$$I(r) := \int_{\mathbb{R}} k(r/s) s^{1/p} \kappa(s)\, \frac{ds}{|s|}.$$

Inserting (2.8) into (2.11) we obtain

$$\|h(r, x_3); L^p_{x_3}(\mathbb{R})\| \leq C r^{\beta+1+\sigma-2\gamma} I(r). \tag{2.12}$$

The assumption $q \in B^{\sigma,p}(\mathbb{R})$ implies that $\kappa \in L^p(\mathbb{R})$ and $\|\kappa; L^p(\mathbb{R})\| \leq C_2\|q; B^{\sigma,p}(\mathbb{R})\|$, for some $C_2 > 0$ independent of q. For $\sigma < 1$ this is a direct implication, otherwise we use Theorem 2.5.1 of [16]. Moreover, we readily check that $k \in L^1(\mathbb{R}^+, \frac{dt}{t})$ (this is the space of integrable functions with respect to the measure $\frac{dt}{t}$) iff $-1 < \sigma < 2\gamma - 1$; therefore Young's theorem leads to

$$\left(\int_0^{+\infty} |I(r)|^p\, \frac{dr}{r}\right)^{1/p} \leq C\|\kappa; L^p(\mathbb{R})\| \leq CC_2\|q; B^{\sigma,p}(\mathbb{R})\|. \tag{2.13}$$

Integrating the p-th power of the estimate (2.12) with respect to r on $]0, 1[$ and using (2.13), we arrive at (2.10). □

We are now able to prove some anisotropic regularities:

Theorem 2.3 *If* $0 < \frac{j\pi}{\omega} < 2 - \frac{2}{p}$*, then*

$$\begin{aligned} \partial_{33} v_j &\in L^p(D), & (2.14)\\ \partial_3 v_j &\in L^p(D), & (2.15)\\ v_j &\in L^p(D), & (2.16)\end{aligned}$$

with norms depending continuously on the L^p*-norm of* g*. If moreover,* $1 - \frac{2}{p} < \frac{j\pi}{\omega}$*, then*

$$\begin{aligned} \partial_{13} v_j, \partial_{23} v_j &\in L^p(D), & (2.17)\\ r^{\gamma-1}\partial_1 v_j, r^{\gamma-1}\partial_2 v_j &\in L^p(D), & (2.18)\\ r^{\gamma-2} v_j &\in L^p(D), & (2.19)\\ r^{-1}\partial_3 v_j &\in L^p(D), & (2.20)\end{aligned}$$

with $\gamma > 2 - \frac{2}{p} - \frac{j\pi}{\omega}$*, the norms depending continuously on the* L^p*-norm of* g*.*

Proof If

$$1 - \frac{2}{p} < \frac{j\pi}{\omega} < 2 - \frac{2}{p}, \tag{2.21}$$

we use Lemma 2.2, with $K(r, x_3) = r^{j\pi/\omega}\partial_{33} K_j(r, x_3)$, since

$$\partial_{33} v_j = (K \overset{x_3}{\star} q_j)(r, x_3)\, \xi(r) \sin\left(\frac{j\pi\varphi}{\omega}\right).$$

This kernel satisfies $|K(r,x_3)| \le C r^{1+j\pi/\omega}(r^2+x_3^2)^{-2}$ for all $r>0, x_3 \in \mathbb{R}$. Therefore, we can apply Lemma 2.2 with $\beta = 1+\frac{j\pi}{\omega}$, $\gamma = 2$ and $\sigma = 2-\frac{2}{p}-\frac{j\pi}{\omega}$ ($\sigma < 1$ due to assumption (2.21)). Since the hypotheses of that theorem are satisfied, estimate (2.10) can be rephrased as

$$\int_0^1 \int_{\mathbb{R}} |\partial_{33} v_j|^p \; r dr dx_3 \le C \, \|q_j; B^{\sigma,p}(\mathbb{R})\|^p.$$

An integration with respect to φ leads to (2.14) and the continuous dependence on the L^p-norm of g.

Conversely, if

$$0 < \frac{j\pi}{\omega} \le 1 - \frac{2}{p}, \tag{2.22}$$

then we know that $q_j \in B^{\sigma',p}(\mathbb{R})$, with $\sigma' = 2-\frac{2}{p}-\frac{j\pi}{\omega} \ge 1$. If $\sigma' = 1$, we use Lemma 2.2 as above with $K(r,x_3) = r^{j\pi/\omega}\partial_{33}K_j(r,x_3)$, $q = q_j \in B^{\sigma,p}(\mathbb{R})$, when $\sigma = \sigma'$. On the contrary, if $\sigma' > 1$, we apply Lemma 2.2 with $K(r,x_3) = r^{j\pi/\omega}\partial_3 K_j(r,x_3)$, $q = \partial_3 q_j \in B^{\sigma,p}(\mathbb{R})$, when $\sigma = \sigma' - 1$.

Analogously, we can consider other derivatives. □

Let us remark that we cannot improve the conclusions of Theorem 2.3. Indeed, when we apply Lemma 2.2, we get an equality in the condition $\beta \ge -1-\frac{2}{p}-\sigma+2\gamma$, in other words we cannot decrease the value of β. This means that, in general, we cannot decrease the power in r in front of the considered derivatives.

Let us also show that the condition $1-\frac{2}{p} < \frac{j\pi}{\omega}$ in the second part of Theorem 2.3 is necessary, in the sense that without this condition, the conclusion could fail.

Lemma 2.4 *If $0 < \frac{j\pi}{\omega} \le 1-\frac{2}{p}$ and $q_j \in B^{2-\frac{2}{p}-\frac{j\pi}{\omega},p}(\mathbb{R})$ is a continuous function such that $q_j \ge 0$, $q_j \not\equiv 0$. Then v_j given by (2.7) satisfies*

$$\frac{1}{r}\partial_\varphi v_j \notin L^p(D). \tag{2.23}$$

Proof By a direct computation, we show that

$$\frac{1}{r}\partial_\varphi v_j = \frac{2j}{\omega} h(r,x_3)\xi(r) r^{-1+j\pi/\omega} \cos(\frac{j\pi\varphi}{\omega}), \tag{2.24}$$

where we have set

$$h(r,x_3) := \int_{\mathbb{R}} \frac{q_j(x_3-rt)}{(1+t^2)^2}\, dt.$$

Since $q_j \not\equiv 0$, there exist $z_0 \in \mathbb{R}$, $\varepsilon > 0$, $\delta > 0$ such that $q_j(x_3) > \delta$ for all $x_3 \in]z_0-\varepsilon, z_0+\varepsilon[$. This implies that for all $x_3 \in]z_0-\varepsilon/2, z_0+\varepsilon/2[$, we have $h(r,x_3) \ge \delta\rho$ for all $r<1$, with $\rho = \int_{-\varepsilon/2}^{\varepsilon/2}(1+t^2)^{-2}\, dt > 0$. Inserting this estimate into (2.24), we get

$$\left|\frac{1}{r}\partial_\varphi v_j\right| \ge \rho' \xi(r) r^{-1+j\pi/\omega} \left|\cos\left(\frac{j\pi\varphi}{\omega}\right)\right|,$$

with some positive constant ρ'. This leads to the conclusion (2.23) because $r^{-1+j\pi/\omega}$ does not belong to L^p with respect to the measure rdr near 0. □

Let us now come back to problem (2.1):

Theorem 2.5 *Let $u \in \overset{o}{H}{}^1(\Omega)$ be the solution of $-\Delta u = f$, with $f \in L^p(\Omega)$. If $p < 6$, then*

$$u \in A^{2,p}_\beta(\Omega) \text{ with } \begin{cases} \beta > 2-\frac{2}{p}-\frac{\pi}{\omega} & \text{for } 2-\frac{2}{p} \ge \frac{\pi}{\omega} > 1-\frac{2}{p}, \\ \beta = 0 & \text{for } 2-\frac{2}{p} < \frac{\pi}{\omega}, \end{cases}$$

and

$$\|u; A^{2,p}_\beta(\Omega)\| \le C\|f; L^p(\Omega)\|.$$

Proof Set $\Omega' = G\times]-z_0, 2z_0[$ and let $w \in \overset{o}{H}{}^1(\Omega')$ (resp. $g \in L^p(\Omega')$) be an odd extension of u to Ω' (resp.f). Then w satisfies $-\Delta w = g$ in Ω'. Fix a cut-off function $\eta \equiv \eta(x_3)$ which equals to 1 on $\bar{\Omega}$ and with support included in Ω'. Using finite differences in the edge direction and Hölder's inequality, we can show that $\eta\partial_3 w \in L^p(\Omega')$, if $p < 6$. Consequently, $v = \eta w \in \overset{o}{H}{}^1(D)$ has a compact support and satisfies $-\Delta v \in L^p(D)$. Using Theorem 2.3 we get $v \in A^{2,p}_\beta(D)$. The restriction to Ω yields the assertion. □

A discussion of other boundary conditions is postponed to Section 4.1.

3 Interpolation error estimates

3.1 The mesh

Recall first the standard mesh grading for two-dimensional corner problems [5, 6, 11, 13]. Let $\overline{G} = \bigcup_i \overline{G}_i$ be a regular triangulation of G. With h being the mesh parameter, $\mu \in]0, 1]$ being the grading parameter, r_i being the distance of G_i to the corner,

$$r_i := \min_{(x_1,x_2)\in\overline{G}_i} (x_1^2 + x_2^2)^{1/2},$$

and some constant $R > 0$, we define real numbers h_i $(i = 1, \ldots, m)$

$$h_i := \begin{cases} h^{1/\mu} & \text{for } r_i = 0, \\ h r_i^{1-\mu} & \text{for } 0 < r_i \le R, \\ h & \text{for } r_i > R, \end{cases} \tag{3.1}$$

and assume that the element size $\mathrm{diam}(G_i)$ is equivalent to h_i.

This kind of grading is now extended in the third direction using a uniform mesh size h. In this way we can get a pentahedral (prismatic elements with triangular basis) or, by dividing each pentahedron, a tetrahedral triangulation $\mathcal{T}_h = \{\Omega_i\}_{i=1}^m$ of Ω, see Figure 4.1 for an illustration. Note that the number m of elements is of the order h^{-3}.

We remark that the elements of such a triangulation satisfy the conditions for which anisotropic error estimates can be derived [1], namely a maximal angle condition and a condition which relates the stretching direction of the elements to the global coordinate system, see also [2] for a detailed explanation.

For a simplified presentation we denote by $h_{i,j}$, $j = 1, \ldots, 3$, the diameter of the element Ω_i in direction x_j, that means, the relations

$$\begin{array}{rcccl} C_1 h_i & \le & h_{j,i} & \le & C_2 h_i, \quad j = 1, 2, \\ C_1 h & \le & h_{3,i} & \le & C_2 h, \end{array} \tag{3.2}$$

are satisfied for $i = 1, \ldots, m$.

We introduce now the finite element space V_h of all continuous functions whose restriction to any Ω_i $(i = 1, \ldots, m)$ is a polynomial of first degree, with the obvious modification to bilinear functions in the pentahedral case. Furthermore, we let V_{0h} be defined by $V_{0h} := \{v_h \in V_h : v_h|_{\partial\Omega} = 0\}$. Note that $V_h \subset H^1(\Omega)$ and $V_{0h} \subset \overset{o}{H}{}^1(\Omega)$. The finite element solutions of problems (2.1) is defined by:

$$\textit{Find } u_h \in V_{0h} \textit{ such that } a(u_h, v_h) = (f, v_h) \textit{ for all } v_h \in V_{0h}. \tag{3.3}$$

The assumptions of the Lax–Milgram lemma are fulfilled; thus this problem has a unique solution.

3.2 Local error estimates in weighted Sobolev spaces

As motivated in the Introduction, we are interested in local interpolation error estimates for anisotropic elements. In [1], the case of classical Sobolev spaces was treated. In this subsection, we shall extend these results to weighted Sobolev spaces and consider particularly the three-dimensional case (tetrahedra and pentahedra). We remark that interpolation error estimates for functions from weighted Sobolev spaces were already proved in [12] for the two-dimensional isotropic case.

We consider first estimates on a reference element $\Omega_0 \in \mathcal{R}$ where $\mathcal{R}$ is the set of reference elements discussed later, see Figures 3.1 and 3.2. We notice here that the elements of $\mathcal{R}$ have the following essential property (P):

(P) For each axis x_i $(i = 1, \dots, 3)$ of the coordinate system there exists one edge E_i of the reference element, which is parallel to this axis and, for normalization, which has length $\text{meas}_1(E_i) = 1$.

Using a similar notation as in [1, §2] we denote by P a space of polynomials, and since each monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ can be identified with the multi-index $\alpha \in \mathbb{N}^3$, we also identify P with the corresponding set of multi-indices. The hull $\overline{P}$ of P is the set $\overline{P} := P \cup \{\alpha + e_i : \alpha \in P,\, i = 1,2,3\}$ ($\{e_i\}_{i=1}^3$ denotes the canonical basis of $\mathbb{R}^3$) and the boundary $\partial\overline{P}$ of P is the set $\overline{P} \setminus P$. Note that $\max_{\alpha\in\overline{P}} |\alpha| = 1 + \max_{\alpha\in P} |\alpha|$.

We introduce now weighted Sobolev spaces on Ω_0: For a finite set $P \subset \mathbb{N}^3$ with $0 \in P$ and for $\beta \in \mathbb{R}$ we set $V_\beta^{P,p}(\Omega_0) := \{v \in \mathcal{D}'(\Omega_0) : \|v; V_\beta^{P,p}(\Omega_0)\| < \infty\}$, where

$$\|v; V_\beta^{P,p}(\Omega_0)\|^p := \sum_{\alpha\in P} \int_{\Omega_0} |r^{\beta-k+|\alpha|} D^\alpha v|^p \, d\underline{x},$$

$k := \max_{\alpha\in P} |\alpha|$, $D^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$, and $r(\underline{x}) := (x_1^2 + x_2^2)^{1/2}$. For $v \in V_\beta^{\overline{P},p}(\Omega_0)$ we also introduce the seminorm

$$|v; V_\beta^{\overline{P},p}(\Omega_0)|^p := \sum_{\alpha\in\partial\overline{P}} \int_{\Omega_0} |r^{\beta-k-1+|\alpha|} D^\alpha v|^p \, d\underline{x}.$$

The space $W^{P,p}(\Omega_0)$ is introduced in analogy to $V_\beta^{P,p}(\Omega_0)$ by omitting the weight.

Lemma 3.1 *Let $P \in \mathbb{N}^3$, P finite with $0 \in P$. Then we have the compact embedding*

$$V_\beta^{\overline{P},p}(\Omega_0) \overset{c}{\hookrightarrow} V_\beta^{P,p}(\Omega_0).$$

Proof For any $v \in V_\beta^{\overline{P},p}(\Omega_0)$ and any fixed $\alpha \in P$, we have

$$\begin{aligned} r^{\beta-k-1+|\alpha|} D^\alpha v &\in L^p(\Omega_0), \\ r^{\beta-k+|\alpha|} D^{\alpha+e_i} v &\in L^p(\Omega_0),\ i = 1,2,3. \end{aligned}$$

This implies $r^{\beta-k+|\alpha|} D^\alpha v \in W^{1,p}(\Omega_0)$, since $|r^{\beta-k+|\alpha|} D^\alpha v| \le C |r^{\beta-k+|\alpha|-1} D^\alpha v|$ almost everywhere in Ω_0. Thus there is a constant $C > 0$ such that

$$\|r^{\beta-k+|\alpha|} D^\alpha v; W^{1,p}(\Omega_0)\| \le C \|v; V_\beta^{\overline{P},p}(\Omega_0)\|. \tag{3.4}$$

Let $\{v_m\}_{m\in\mathbb{N}}$ be a sequence in $V_\beta^{\overline{P},p}(\Omega_0)$ such that for some $K > 0$ and for all $m \in \mathbb{N}$ the relation $\|v_m; V_\beta^{\overline{P},p}(\Omega_0)\| < K$ holds. From (3.4) we obtain for all $m \in \mathbb{N}$ and $\alpha \in P$ the bound $\|r^{\beta-k+|\alpha|} D^\alpha v_m; W^{1,p}(\Omega_0)\| \le C$. Owing to the compact embedding $W^{1,p}(\Omega_0) \overset{c}{\hookrightarrow} L^p(\Omega_0)$ (Rellich–Kondrašov theorem), there is a subsequence $\{v_{m_k}\}$ such that for all $\alpha \in P$

$$r^{\beta-k+|\alpha|} D^\alpha v_{m_k} \to w_\alpha \text{ in } L^p(\Omega_0). \tag{3.5}$$

(Since P is finite we can use card(P) times this theorem.) Because $0 \in P$ we obtain in particular

$$r^{\beta-k}v_{m_k} \to w_0 := r^{\beta-k}v \in L^p(\Omega_0),$$

which implies $v_{m_k} \to v$ in $\mathcal{D}'(\Omega_0)$, and $D^\alpha v_{m_k} \to D^\alpha v$ in $\mathcal{D}'(\Omega_0)$ for all $\alpha \in P$. With (3.5), we deduce that $w_\alpha = r^{\beta-k+|\alpha|}D^\alpha v \in L^p(\Omega_0)$ and therefore $v_{m_k} \to v$ in $V_\beta^{P,p}(\Omega_0)$. Thus the embedding is proved. □

We show now that, under some condition on β, elements of $V_\beta^{P,p}(\Omega_0)$ are in $L^1(\Omega_0)$, as well as all derivatives with respect to P.

Lemma 3.2 *Let $P \subset \mathbb{N}^3$, P finite, such that $0 \in P$. If $\beta < 2 - \frac{2}{p}$ then for all $v \in V_\beta^{P,p}(\Omega_0)$ the following relation holds:*

$$D^\alpha v \in L^1(\Omega_0) \text{ for all } \alpha \in P. \tag{3.6}$$

Proof If $\beta \le 0$ the assertion is obvious since $V_\beta^{P,p}(\Omega_0) \hookrightarrow W^{P,p}(\Omega_0)$. If $\beta > 0$, then we have $r^{\beta-k+|\alpha|}D^\alpha v \in L^p(\Omega_0)$ for any $\alpha \in P$. Since $|\alpha| \le k$ we deduce that $r^\beta D^\alpha v \in L^p(\Omega_0)$. Using Hölder's inequality, we show that this implies (3.6): Indeed, we have for $\frac{1}{p} + \frac{1}{q} = 1$

$$\int_{\Omega_0} |D^\alpha v|\, d\underline{x} = \int_{\Omega_0} r^{-\beta}|r^\beta D^\alpha v|\, d\underline{x} \le \|r^{-\beta}; L^q(\Omega_0)\|\, \|r^\beta D^\alpha v; L^p(\Omega_0)\|.$$

The $L^q(\Omega_0)$-norm of $r^{-\beta}$ is finite if and only if $\beta q < 2$ (by using cylindrical coordinates (r, φ, z)). But this is equivalent to $\beta < 2 - \frac{2}{p}$. □

From Lemmas 3.1 and 3.2 and using the same arguments as in [1, Lemma 2], we obtain the following lemma.

Lemma 3.3 *Let $P \in \mathbb{N}^3$ be a finite set of multi-indices with $0 \in P$. If $\beta < 2 - \frac{2}{p}$ then there is a constant $C > 0$ such that*

$$\|v; V_\beta^{\overline{P},p}(\Omega_0)\| \le C|v; V_\beta^{\overline{P},p}(\Omega_0)| \tag{3.7}$$

for all $v \in V_\beta^{\overline{P},p}(\Omega_0)$ satisfying $\int_{\Omega_0} D^\alpha v\, d\underline{x} = 0$ for $\alpha \in P$.

We are now ready to give the interpolation estimate, first in a very general form, then especially for our purposes.

Lemma 3.4 *Let $\beta < 2 - \frac{2}{p}$ be a real number, and let $P, Q \subset \mathbb{N}^3$ and $\gamma \in \mathbb{N}^3$ be such that $0 \in Q$ and $Q + \gamma \subset P$. Further introduce a linear operator $I : C^\mu(\Omega_0) \to P$, $\mu \in \mathbb{N}$, and assume that there are linear functionals $F_i \in \left(V_\beta^{\overline{Q},p}(\Omega_0)\right)'$, $i = 1, \dots, j$, $j = \dim D^\gamma P$, satisfying*

$$\begin{aligned} &F_i(D^\gamma I v) = F_i(D^\gamma v) \quad (i = 1, \dots, j) \text{ for all } v \in C^\mu(\Omega_0) \cap V_\beta^{\overline{Q}+\gamma,p}(\Omega_0), \\ &F_i(D^\gamma q) = 0 \text{ for all } i = 1, \dots, j \Longrightarrow D^\gamma q = 0 \text{ for all } q \in P. \end{aligned} \tag{3.8}$$

Then there is a constant $C > 0$ such that

$$\|D^\gamma(v - Iv); V_\beta^{\overline{Q},p}(\Omega_0)\| \le C|D^\gamma v; V_\beta^{\overline{Q},p}(\Omega_0)|$$

for all $v \in C^\mu(\Omega_0) \cap V_\beta^{\overline{Q}+\gamma,p}(\Omega_0)$.

Proof We follow the proof of Lemma 3 of [1], since Lemma 1 of [1] can be extended to the spaces $V_\beta^{P,p}(\Omega_0)$ (owing to Lemma 3.2), while Lemma 2 of [1] is replaced by Lemma 3.3. □

Theorem 3.5 *Suppose that $0 \le \beta < 1 - \frac{1}{p}$, $p > 2$, and let Iv be the (bi-)linear Lagrangian interpolant of v with respect to the vertices. Then for all $v \in A^{2,p}_\beta(\Omega_0) \cap C(\Omega_0)$ we have for $i = 1, 2$*

$$\left\| r^{\beta-1}\partial_i(v - Iv); L^p(\Omega_0)\right\| \le C\left\{\int_{\Omega_0}\left[r^{p\beta}\left(|\partial_{1i}v|^p + |\partial_{2i}v|^p\right) + |\partial_{3i}v|^p\right] d\underline{x}\right\}^{1/p}, \quad (3.9)$$

$$\left\| r^{-1}\partial_3(v - Iv); L^p(\Omega_0)\right\| \le C\left\{\int_{\Omega_0}\left(|\partial_{13}v|^p + |\partial_{23}v|^p + |\partial_{33}v|^p\right) d\underline{x}\right\}^{1/p}. \quad (3.10)$$

Proof We set $Q := \{(0,0,0)\}$, $\overline{Q} := \{(0,0,0)\} \cup \{e_i\}_{i=1,2,3}$ and remark that $v \in A^{2,p}_\beta(\Omega_0)$ implies $\partial_i v \in V^{1,p}_\beta(\Omega_0) = V^{\overline{Q},p}_\beta(\Omega_0)$ $(i = 1, 2)$ and $\partial_3 v \in V^{1,p}_0(\Omega_0) = V^{\overline{Q},p}_0(\Omega_0)$. To prove the assertion we apply Lemma 3.4 with $P = \overline{Q}$, $\gamma := e_i$ and $F_1(v) := \int_{E_i} v\,dx_i$, where E_i is that edge of Ω_0 which is parallel to the x_i-axis, see Property (P) on page 8. It remains to prove the continuity of F_1.

In the simpler case $i = 3$ we can use the embeddings

$$V^{1,p}_0(\Omega_0) \hookrightarrow W^{1,p}(\Omega_0) \hookrightarrow W^{1-2/p,p}(E_3) \hookrightarrow L^1(E_3)$$

which holds for $1 - \frac{2}{p} > 0$, that means $p > 2$.

For $i = 1, 2$ we use that $v \in V^{1,p}_\beta(\Omega_0)$ implies

$$r^\beta v \in W^{1,p}(\Omega_0) \hookrightarrow W^{1-2/p,p}(E_i) \hookrightarrow L^p(E_i), \; i = 1, 2.$$

Using Hölder's inequality we conclude for $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\int_{E_i} |v|\,dx_i \le \|r^{-\beta}; L^q(E_i)\|\,\|r^\beta v; L^p(E_i)\| \le \|r^{-\beta}; L^q(E_i)\|\,\|v; V^{1,p}_\beta(\Omega_0)\|.$$

Using that $r^{-\beta} \in L^q(E_i)$ for $\beta < \frac{1}{q} = 1 - \frac{1}{p}$ the proof is complete. □

Remark 3.6 In applications with the same type of boundary conditions on both faces of the edge, we have $\beta = 2 - \frac{2}{p} - \frac{\pi}{\omega} + \varepsilon$ with an arbitrarily small positive real ε. That means $\beta < 1 - \frac{1}{p}$ is equivalent to $1 - \frac{1}{p} < \frac{\pi}{\omega}$, so that for p close to 2 this condition always holds.

Corollary 3.7 *For $p > 2$, $0 \le \beta < 1 - \frac{1}{p}$, we have for $v \in A^{2,p}_\beta(\Omega_0)$ and $i = 1, 2$ the estimates*

$$\|\partial_i(v - Iv); L^p(\Omega_0)\| \le C\left\{\int_{\Omega_0}\left[r^{p\beta}\left(|\partial_{1i}v|^p + |\partial_{2i}v|^p\right) + |\partial_{3i}v|^p\right] d\underline{x}\right\}^{1/p}, \quad (3.11)$$

$$\|\partial_3(v - Iv); L^p(\Omega_0)\| \le C\left\{\int_{\Omega_0}\left(|\partial_{13}v|^p + |\partial_{23}v|^p + |\partial_{33}v|^p\right) d\underline{x}\right\}^{1/p}. \quad (3.12)$$

Proof The assertion follows from (3.9) and (3.10) since the weights on the left hand side are bounded from below by some constant $C > 0$. □

Now we are going to transform these estimates to the actual finite elements Ω_i. We realize that for two tetrahedra/pentahedra Ω_i and Ω_0 there is an affine linear transformation

$$\underline{x} = F(\underline{y}) = B\underline{y} + \underline{b} \quad (3.13)$$

with $B = (b_{jk})^3_{j,k=1} \in \mathbb{R}^{3\times3}$, $\underline{b} = (b_j)^3_{j=1} \in \mathbb{R}^3$, such that $\Omega_i = F(\Omega_0)$. For pentahedra one reference element is sufficient, for example $\Omega_0 := \{\underline{y} \in \mathbb{R}^3 : y_1 > 0,\; y_2 > 0,\; y_1 + y_2 < 1,\; 0 < y_3 < 1\}$. In the tetrahedral case we consider two reference elements Ω_a and Ω_b as given in Figure 3.1. Note that anisotropic tetrahera can have three or four edges with length of order h_3, they are mapped to Ω_a and Ω_b, respectively.

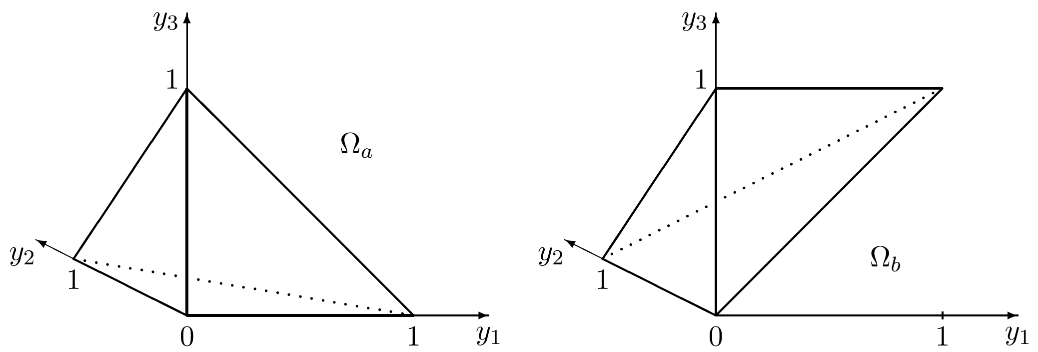

Figure 3.1: Basic reference elements for anisotropic interpolation error estimates in the three-dimensional case.

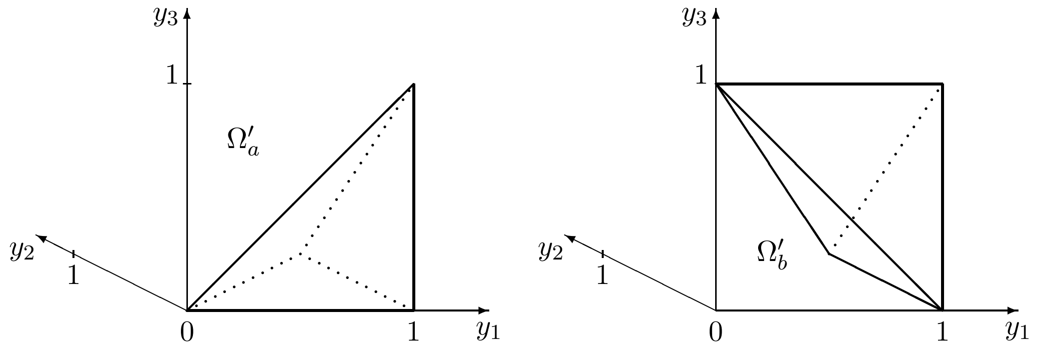

Figure 3.2: Additional reference elements for interpolation error estimates in weighted Sobolev spaces.

We see that in our case (compare Section 3.1) $b_{13} = b_{23} = b_{31} = b_{32} = 0$ and the submatrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ satisfies the properties which are well-known from the two-dimensional isotropic case. This leads to the following relations for the matrix elements b_{jk} and $b_{jk}^{(-1)}$ of B and B^{-1}, respectively:

$$|b_{jk}| \leq C \min\{h_{j,i}, h_{k,i}\}, \quad |b_{jk}^{(-1)}| \leq C \min\{h_{j,i}^{-1}, h_{k,i}^{-1}\}. \tag{3.14}$$

Using Corollary 3.7 for the special case $\beta = 0$ we get with (3.14) the estimate [1]

$$|v - Iv; W^{1,p}(\Omega_i)| \leq C \sum_{k=1}^{3} h_{k,i} |\partial_k v; W^{1,p}(\Omega_i)| \quad \text{for } p > 2. \tag{3.15}$$

To transform the estimates (3.11), (3.12) for $\beta > 0$ we can assume that h_1 and h_2 are of the same order, but we need additionally that

$$y_1^2 + y_2^2 \leq C h_1^{-2} (x_1^2 + x_2^2) \quad \text{for all } \underline{x} \in \Omega_i, \tag{3.16}$$

which can be concluded from $b_{13} = b_{23} = 0$ and $b_1 = b_2 = 0$. The geometrical meaning of the latter condition is that at least one vertex of Ω_i is contained in the x_3-axis, and that this vertex is mapped to the corresponding vertex at the y_3-axis. For treating all possible cases, we have to extend the set of reference elements in the tetrahedral case to $\mathcal{R} = \{\Omega_a, \Omega'_a, \Omega_b, \Omega'_b\}$, where Ω'_a and Ω'_b are obtained from Ω_a and Ω_b, respectively, by a reflection at the plane $y_1 = \frac{1}{2}$, see Figure 3.2. One can choose the appropriate reference element by the number of edges with length of order h_3 (three or four) and the number of vertices of Ω_i that are contained in the x_3-axis (one or two).

Theorem 3.8 *Let $I_h v$ be the (bi-)linear Lagrangian interpolant of $v \in A^{2,p}_\beta(\Omega_i)$ with respect to the vertices. Assume further that at least one vertex of element Ω_i is located at th x_3-axis. Then for $v \in A^{2,p}_\beta(\Omega_i)$, $0 \le \beta < 1 - \frac{1}{p}$, $p > 2$, the norm of the derivatives of the interpolation error can be estimated by*

$$\begin{aligned} &\|\partial_j(v - Iv); L^p(\Omega_i)\| \\ &\quad\le\; C\left\{\int_{\Omega_i}\left[h_{1,i}^{p(1-\beta)} r^{p\beta}\left(|\partial_{1j}v|^p + |\partial_{2j}v|^p\right) + h_{3,i}^p|\partial_{3j}v|^p\right] d\underline{x}\right\}^{1/p}, \quad i = 1,2, \end{aligned} \tag{3.17}$$

$$\|\partial_3(v - Iv); L^p(\Omega_i)\| \le C\left\{\int_{\Omega_i}\sum_{k=1}^{3} h_{k,i}^p\,|\partial_{k3}v|^p\, d\underline{x}\right\}^{1/p}. \tag{3.18}$$

Proof The assertion is a direct consequence from Corollary 3.7 using the transformation (3.13) with (3.14) and (3.16). □

Corollary 3.9 *Under the assumptions of Theorem 3.8 the following estimate holds:*

$$|v - I_h v; W^{1,p}(\Omega_i)| \le C\,(h_{1,i}^{1-\beta} + h_{3,i})|v; A^{2,p}_\beta(\Omega_i)|. \tag{3.19}$$

3.3 Global error estimates

In this section, we investigate first the global interpolation error, that is the difference between the solution u of our boundary value problem (2.3) and its piecewise (bi-)linear interpolant $I_h u$ on the family of anisotropically graded meshes introduced in Subsection 3.1. The difficulty is that we are interested on one hand in an estimate in the energy norm which is equivalent to $|\,.\,; W^{1,2}(\Omega)|$, in order to apply Céa's lemma for the finite element error. But on the other hand the local interpolation error estimates (3.15) and (3.19) are valid for $|\,.\,; W^{1,p}(\Omega_i)|$ with $p > 2$ only.

Theorem 3.10 *Let u be the solution of the boundary value problem (2.3) with $f \in L_p(\Omega)$, $2 < p < p_+$,*

$$p_+ := \min\left\{6; \left(1 - \frac{\pi}{\omega}\right)^{-1}\right\}. \tag{3.20}$$

Then for the interpolation error $u - I_h u$ the following estimate holds:

$$|u - I_h u; W^{1,2}(\Omega)| \le C h^s \|f; L^p(\Omega)\|,$$

$$s = \begin{cases} 1 & \text{for } \mu < \frac{\pi}{\omega}\cdot\frac{p}{2p-2}, \\ \frac{2}{p} - 1 + \frac{1}{\mu}\cdot\frac{\pi}{\omega} - \varepsilon & \text{for } \mu \ge \frac{\pi}{\omega}\cdot\frac{p}{2p-2}. \end{cases} \tag{3.21}$$

Proof We reduce the estimation of the global error to the evaluation of the local errors and distinguish between the $m_0 = \mathcal{O}(h^{-1})$ elements whose closure has at least one common point with the edge, and the $m - m_0 = \mathcal{O}(h^{-3})$ elements away from the edge:

$$|u - I_h u; W^{1,2}(\Omega)|^2 = \sum_{i=1}^{m_0}|u - I_h u; W^{1,2}(\Omega_i)|^2 + \sum_{i=m_0+1}^{m}|u - I_h u; W^{1,2}(\Omega_i)|^2. \tag{3.22}$$

For the elements in the first sum we apply the local estimate (3.19). Using Hölder's inequality, we have for $i = 1, \ldots, m_0$

$$\begin{aligned} |u - I_h u; W^{1,2}(\Omega_i)|^p &\le (\mathrm{meas}\Omega_i)^{-1+p/2}|u - I_h u; W^{1,p}(\Omega_i)|^p \\ &\le C(hh_i^2)^{-1+p/2}(h_i^{1-\beta} + h)^p|u; A^{2,p}_\beta(\Omega_i)|^p. \end{aligned}$$

Summing up these estimates for all $i = 1, \dots, m_0$, and using again Hölder's inequality, we can conclude

$$\begin{aligned}
\sum_{i=1}^{m_0} |u - I_h u; W^{1,2}(\Omega_i)|^2 &\le m_0^{1-2/p} \left(\sum_{i=1}^{m_0} |u - I_h u; W^{1,p}(\Omega_i)|^p \right)^{2/p} \\
&\le C \sum_{i=1}^{m_0} h^{-1+2/p} (h h_i^2)^{1-2/p} (h_i^{1-\beta} + h)^2 |u; A_\beta^{2,p}(\Omega_i)|^2 \\
&\le C \left(h^{(2-\beta-2/p)/\mu} + h^{1+(1-2/p)/\mu} \right)^2 \|f; L^p(\Omega)\|^2.
\end{aligned}$$

Since for $\beta = \max\{0;\ 2 - \frac{2}{p} - \frac{\pi}{\omega} + \varepsilon'\}$ there holds $\frac{1}{\mu}(2 - \frac{2}{p} - \beta) > s$, and we have directly $1 + \frac{1}{\mu}(1 - \frac{2}{p}) > 1 \ge s$ (with s from (3.21)), we get

$$\sum_{i=1}^{m_0} |u - I_h u; W^{1,2}(\Omega_i)|^2 \le C h^{2s} \|f; L^p(\Omega)\|^2. \tag{3.23}$$

For the elements in the second sum of (3.22) we can use that $u \in W^{2,p}(\Omega_i)$, $i = m_0 + 1, \dots, m$, and thus apply the local estimate (3.15). Again with Hölder's inequality, we have for $i = m_0 + 1, \dots, m$:

$$\begin{aligned}
&|u - I_h u; W^{1,2}(\Omega_i)|^p \le (\mathrm{meas}\Omega_i)^{-1+p/2} |u - I_h u; W^{1,p}(\Omega_i)|^p \\
&\le C (h h_i^2)^{-1+p/2} \left(h_i^p \sum_{\substack{|\alpha|=2 \\ \alpha_3=0}} \|D^\alpha u; L^p(\Omega_i)\|^p + C h^p \sum_{\substack{|\alpha|=2 \\ \alpha_3>0}} \|D^\alpha u; L^p(\Omega_i)\|^p \right)
\end{aligned} \tag{3.24}$$

For $\mu < \frac{\pi}{\omega} \cdot \frac{p}{2p-2}$ we can estimate $h_i^{2p-2} \le C h^{2p-2} r_i^{(2p-2)(1-\mu)} = C h^{2p-2} r_i^{p\beta}$ with $\beta = \frac{1}{p}(2p-2)(1-\mu) > 2 - \frac{2}{p} - \frac{\pi}{\omega}$.

For $\mu \ge \frac{\pi}{\omega} \cdot \frac{p}{2p-2}$ we have to use part of h_i^{2p-2} via $h_i < C r_i$ to get also the power $p\beta$ of r_i on the right hand side:

$$\begin{aligned}
h_i^{2p-2} &= h_i^{\frac{p}{\mu}\cdot\frac{\pi}{\omega} - p\varepsilon} h_i^{2p-2-\frac{p}{\mu}\cdot\frac{\pi}{\omega} + p\varepsilon} \\
&\le C h^{\frac{p}{\mu}\cdot\frac{\pi}{\omega} - p\varepsilon} r_i^{(\frac{p}{\mu}\cdot\frac{\pi}{\omega} - p\varepsilon)(1-\mu)} r_i^{2p-2-\frac{p}{\mu}\cdot\frac{\pi}{\omega} + p\varepsilon} \\
&= C h^{\frac{p}{\mu}\cdot\frac{\pi}{\omega} - p\varepsilon} r_i^{p\beta}
\end{aligned}$$

with $\beta = \frac{1}{p}\left[\left(\frac{p}{\mu} \cdot \frac{\pi}{\omega} - p\varepsilon \right)(1-\mu) + 2p - 2 - \frac{p}{\mu} \cdot \frac{\pi}{\omega} + p\varepsilon \right] = 2 - \frac{2}{p} - \frac{\pi}{\omega} + \frac{\varepsilon}{\mu} > 2 - \frac{2}{p} - \frac{\pi}{\omega}$ for $\varepsilon > 0$. Note that $h_i^{2p-2-\frac{p}{\mu}\cdot\frac{\pi}{\omega}+p\varepsilon} < C r_i^{2p-2-\frac{p}{\mu}\cdot\frac{\pi}{\omega}+p\varepsilon}$ because $2p - 2 - \frac{p}{\mu} \cdot \frac{\pi}{\omega} + p\varepsilon > 2p - 2 - \frac{p}{\mu} \cdot \frac{\pi}{\omega} \ge 0$ due to the assumption on μ.

Thus we get with (3.24)

$$|u - I_h u; W^{1,2}(\Omega_i)|^p \le C h^{ps + 3(p-2)/2} \|u; A_\beta^{2,p}(\Omega_i)\|^p$$

with s from (3.21). Summing up these estimates for all $i = m_0 + 1, \dots, m$, and using again Hölder's inequality, we can conclude with Corollary 2.5

$$\begin{aligned}
\sum_{i=m_0+1}^{m} |u - I_h u; W^{1,2}(\Omega_i)|^2 &\le (m - m_0)^{1-\frac{2}{p}} \left(\sum_{i=m_0+1}^{m} |u - I_h u; W^{1,2}(\Omega_i)|^p \right)^{2/p} \\
&\le C h^{-3(1-\frac{2}{p})} h^{\frac{2}{p}(ps + \frac{3}{2}(p-2))} \|u; A_\beta^{2,p}(\Omega)\|^2 \\
&= C h^{2s} \|f; L^p(\Omega)\|^2.
\end{aligned} \tag{3.25}$$

From (3.23) and (3.25) we get the assertion. □

Corollary 3.11 *Let u be the solution of the boundary value problem (2.3) with $f \in L_p(\Omega)$, $2 < p < p_+$, p_+ from (3.20), and let u_h be the finite element solution of (3.3). Then the error estimate*

$$\|u - u_h; W^{1,2}(\Omega)\| \le C|u - u_h; W^{1,2}(\Omega)| \le Ch^s\|f; L^p(\Omega)\|$$

holds, with s from (3.21).

Remark 3.12 Note that the restriction $p < p_+$ is not essential for this estimate, because $f \in L^p(\Omega)$ yields $f \in L^q(\Omega)$ for $q \le p$ and $\|f; L^q(\Omega)\| \le C\|f; L^p(\Omega)\|$. We can apply Theorem 3.10 for $q < p_+$. Nevertheless, we have to replace p in (3.21) by $\min\{p;\ p_+ - \delta\}$, $\delta > 0$ arbitrary.

Remark 3.13 In order to use meshes which are not too much refined, the estimates are most favourable for p close to 2. For $p = 2 + \delta$ (δ is an arbitrarily small real number) we have

$$s = \begin{cases} 1 & \text{for } \mu < \frac{\pi}{\omega}\left(1 - \frac{\delta}{2+2\delta}\right), \\ \frac{1}{\mu}\cdot\frac{\pi}{\omega} - \varepsilon - \frac{\delta}{2+\delta} & \text{for } \mu \ge \frac{\pi}{\omega}\left(1 - \frac{\delta}{2+2\delta}\right), \end{cases}$$

so that one can conclude that the approximation order s is

$$s = \begin{cases} 1 & \text{for } \mu < \frac{\pi}{\omega}, \\ \frac{1}{\mu}\cdot\frac{\pi}{\omega} - \varepsilon & \text{for } \mu \ge \frac{\pi}{\omega}, \end{cases} \tag{3.26}$$

(even when f is smoother). On the other hand it is not clear in which way the constant C depends on p; we suspect that $C \to \infty$ for $p \to 2$.

Remark 3.14 With analogous arguments as in [5] we can prove that the condition number of the stiffness matrix is of order h^{-2}, that means, of the same order as for quasiuniform meshes.

4 Other aspects

4.1 Extension to general boundary conditions

All the results of Sections 2 and 3 can be extended to Neumann boundary conditions by replacing $\sin(\frac{j\pi\varphi}{\omega})$ by $\cos(\frac{j\pi\varphi}{\omega})$ everywhere.

In the case of mixed boundary conditions the analytical results extend with $\sin(\frac{j\pi\varphi}{\omega})$ replaced by $\sin(\frac{(j-\frac{1}{2})\pi\varphi}{\omega})$. But the condition $0 \le \beta < 1 - \frac{1}{p}$ in Theorem 3.8, with $\beta = 2 - \frac{2}{p} - \frac{\pi}{2\omega} + \varepsilon$, leads to the restriction $\omega < \pi$. This restriction is known from the isotropic case (see [5]); it is equivalent to the condition that u must be contained in $W^{3/2+\varepsilon,2}(\Omega) \hookrightarrow C(\overline{\Omega})$ in order to have well-defined pointwise values of u. Only in that case interpolation makes sense.

Newton boundary conditions need more explanation. Consider the problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \underline{n}} + \sigma u = 0 \quad \text{on } \partial\Omega, \tag{4.1}$$

where $\sigma \ge 0$ on $\partial\Omega$, $\sigma(\underline{x}) > \sigma_0 > 0$, for all $\underline{x}$ in a part $\partial\Omega_T \subset \partial\Omega$ with $meas_2 \partial\Omega_T > 0$, and $\sigma \in W^{1-1/p,p}(\partial\Omega) \cap W^{1-1/s,s}(\partial\Omega)$, for some $s > 3$.

Theorem 4.1 *Let $u \in H^1(\Omega)$ be the variational solution of (4.1) with $p < 6$, then*

$$u \in A^{2,p}_\beta(\Omega) \ \textit{with} \ \begin{cases} \beta > 2 - \frac{2}{p} - \frac{\pi}{\omega} & \textit{for } 2 - \frac{2}{p} \ge \frac{\pi}{\omega} > \frac{3}{2} - \frac{3}{p} \\ \beta = 0 & \textit{for } 2 - \frac{2}{p} < \frac{\pi}{\omega} \end{cases}$$

and

$$\|u; A^{2,p}_\beta(\Omega)\| \le C\|f; L^p(\Omega)\|.$$

Proof First we transform the boundary conditions into

$$\frac{\partial u}{\partial \underline{n}} = -\sigma u \text{ on } \partial\Omega,$$

and use a lifting trace theorem in order to come back to homogeneous Neumann boundary conditions. Indeed, since there exists $s > 3$ such that $\sigma \in W^{1-1/s,s}(\partial\Omega)$, there exists $\tilde{\sigma} \in W^{1,s}(\Omega)$ such that $\tilde{\sigma} = \sigma$ on $\partial\Omega$. Using Theorem 1.4.4.2 of [9] we get $\tilde{\sigma} u \in H^1(\Omega)$, because $u \in H^1(\Omega)$. Consequently, $\sigma u \in H^{1/2}(\partial\Omega)$, and because of the classical trace theorem, there exists $w \in H^2(\Omega)$ such that

$$\frac{\partial w}{\partial \underline{n}} = -\sigma u \text{ on } \partial\Omega.$$

This means that $u_1 := u - w \in H^1(\Omega)$ is the solution of

$$\int_\Omega \nabla u_1 \cdot \nabla v \, d\underline{x} = \int_\Omega f_1 v \, d\underline{x} \text{ for all } v \in H^1(\Omega),$$

where $f_1 := f + \Delta w$. Since $f_1 \in L^2(\Omega)$ and owing to Theorem 23.3 of [8], we conclude that

$$u \in H^{1+\pi/\omega-\varepsilon}(\Omega) \text{ for all } \varepsilon > 0.$$

Using again Theorem 1.4.4.2 of [9] to $u \in H^{1+\pi/\omega-\varepsilon}(\Omega)$ and some $\hat{\sigma} \in W^{1,p}(\Omega)$ such that $\hat{\sigma} = \sigma$ on $\partial\Omega$, we obtain that

$$\hat{\sigma} u \in W^{1,p}(\Omega), \text{ if } \frac{\pi}{\omega} > \frac{3}{2} - \frac{3}{p}.$$

Note that the condition $\frac{\pi}{\omega} > \frac{3}{2} - \frac{3}{p}$ is necessary to have the embedding $H^{1+\pi/\omega-\varepsilon}(\Omega) \hookrightarrow W^{1,p}(\Omega)$. With the help of the classical trace theorem, there exists $w_1 \in W^{2,p}(\Omega)$ such that

$$\frac{\partial w_1}{\partial \underline{n}} = -\sigma u \text{ on } \partial\Omega.$$

Finally, setting $u_2 = u - w_1$, we see that $u_2 \in H^1(\Omega)$ and that it is a solution of

$$\int_\Omega \nabla u_2 \cdot \nabla v \, d\underline{x} = \int_\Omega f_2 v \, d\underline{x} \text{ for all } v \in H^1(\Omega),$$

with $f_2 := f + \Delta w_1 \in L^p(\Omega)$. Applying Corollary 2.5 to u_2 (in the case of homogeneous Neumann boundary conditions), we conclude that $u_2 \in A^{2,p}_\beta(\Omega)$ with β satisfying the conditions of that corollary. Since $w_1 \in W^{2,p}(\Omega)$, we get the assertion. □

Thus, Corollary 3.12 applies with the restriction

$$2 < p < p_+ := \min\left\{6;\ (\frac{1}{2} - \frac{\pi}{3\omega})^{-1};\ (1 - \frac{\pi}{\omega})^{-1}\right\}.$$

Note that the proof of this case needs also a proof of $\|u - I_h u; L^2(\Omega)\| \leq Ch^s \|f; L^p(\Omega)\|$ which can be carried out with the same arguments as in the proof of Theorem 3.10.

4.2 Numerical tests

As an example we consider the Poisson problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega^{(1)}, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega^{(2)}, \end{aligned}$$

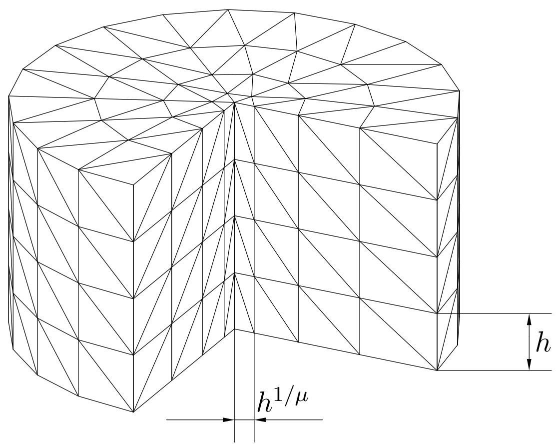

Figure 4.1: Example for an anisotropic mesh.

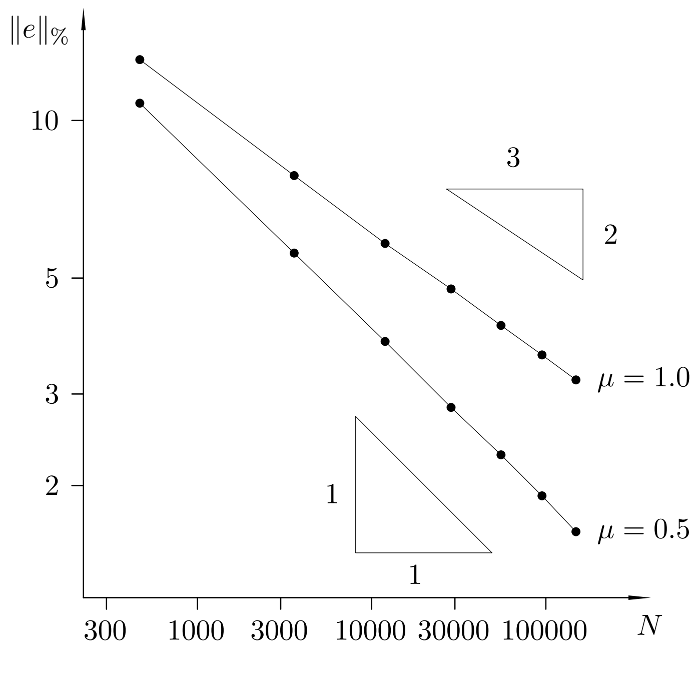

Figure 4.2: Behaviour of the finite element error for ungraded ($\mu = 1.0$) and anisotropically graded meshes ($\mu = 0.5$).

in the three-dimensional domain $\Omega = \{(x_1, x_2, x_3) = (r\cos\varphi, r\sin\varphi, z) \in \mathbb{R}^3 : \ r < 1,\ 0 < \varphi < \frac{3}{2}\pi,\ 0 < z < 1\}$ with the boundary $\partial\Omega = \partial\Omega^{(1)} \cup \partial\Omega^{(2)}$, $\partial\Omega^{(1)} = \{\underline{x} \in \partial\Omega : \ r = 1\}$. The right hand sides f and g are taken such that

$$\begin{aligned} u &= \gamma r^{2/3} \cos \frac{2}{3}\varphi \\ \gamma &= \begin{cases} z^2 + 1 & \text{for } z \in [0, \frac{1}{2}] \\ -z^2 + 2z + \frac{1}{2} & \text{for } z \in (\frac{1}{2}, 1] \end{cases} \end{aligned}$$

is the exact solution of the problem. It has the typical behaviour in the neighbourhood of the edge for a Neumann problem. We remark that $u \in A^{2,p}_{\beta}(\Omega)$ if and only if $\beta > 2 - \frac{2}{p} - \frac{2}{3}$, that $f = -\Delta u \in L_p(\Omega)$ for all $p \in [1, \infty]$, and that f has a jump at $z = \frac{1}{2}$.

The meshes used were constructed as described in Subsection 3.1, see also Figure 4.1. In order to investigate the influence of anisotropic mesh grading on the approximation order we varied the mesh size h $(\frac{1}{6}, \frac{1}{12}, \frac{1}{18}, \ldots, \frac{1}{42})$ and computed numerical solutions for $\mu = 1.0$ and $\mu = 0.5$. From them we calculated the energy norm $\|e\|$ of the finite element error $e = u - u_h$ by numerical integration with an 11-point formula. The relative norms $\|e\|_{\%} := \|e\|/\|u_h\|$ are plotted against the number N of unknowns using a double logarithmic scale in Figure 4.2.

The computations show that the use of anisotropic meshes leads to the optimal approximation order and diminishes the error even for rather coarse meshes.

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References

[1] Th. Apel and M. Dobrowolski. Anisotropic interpolation with applications to the finite element method. *Computing*, 47:277–293, 1992.

[2] Th. Apel and G. Lube. Local inequalities for anisotropic finite elements and their application to convection-diffusion problems. Preprint SPC94_26, TU Chemnitz-Zwickau, 1994.

[3] Th. Apel and F. Milde. Realization and comparison of various mesh refinement strategies near edges. Preprint SPC94_15, TU Chemnitz-Zwickau, 1994.

[4] Th. Apel and S. Nicaise. Elliptic problems in domains with edges: anisotropic regularity and anisotropic finite element meshes. Preprint SPC94_16, TU Chemnitz-Zwickau, 1994.

[5] Th. Apel, A.-M. Sändig, and J. R. Whiteman. Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains. Technical Report 12, BICOM Institute of Computational Mathematics, 1993. To appear in Math. Meth. Appl. Sci.

[6] I. Babuška, R. B. Kellogg, and J. Pitkäranta. Direct and inverse error estimates for finite elements with mesh refinements. *Numer. Math.*, 33:447–471, 1979.

[7] P. Clement. Approximation by finite element functions using local regularization. *RAIRO Anal. Numer.*, 2:77–84, 1975.

[8] M. Dauge. *Elliptic boundary value problems on corner domains*, volume 1341 of *Lecture Notes in Mathematics*. Springer–Verlag, Berlin, 1988.

[9] P. Grisvard. *Elliptic problems in nonsmooth domains.* Pitman, Boston–London–Melbourne, 1985.

[10] P. Grisvard. Singular behaviour of elliptic problems in non Hilbertian Sobolev spaces. *J. Math. Pures Appl.*, 74:3–33, 1995.

[11] L. A. Oganesyan and L. A. Rukhovets. *Variational-difference methods for the solution of elliptic equations.* Izd. Akad. Nauk Armyanskoi SSR, Jerevan, 1979. (Russian).

[12] G. Raugel. *Résolution numérique de problèmes elliptiques dans des domaines avec coins.* PhD thesis, Université de Rennes (France), 1978.

[13] G. Raugel. Résolution numérique par une méthode d'éléments finis du problème Dirichlet pour le Laplacien dans un polygone. *C. R. Acad. Sci. Paris, Sér. A*, 286(18):A791–A794, 1978.

[14] L. R. Scott and S. Zhang. Finite element interpolation of non-smooth functions satisfying boundary conditions. *Math. Comp.*, 54:483–493, 1990.

[15] E. Stephan and J. R. Whiteman. Singularities of the Laplacian at corners and edges of three–dimensional domains and their treatment with finite element methods. *Math. Meth. Appl. Sci.*, 10(3):339–350, 1988.

[16] H. Triebel. *Interpolation theory, function spaces, differential operators.* North-Holland, Amsterdam, 1978.